

# NULL CONTROLLABILITY OF A CASCADE MODEL IN POPULATION DYNAMICS

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**ABSTRACT.** In this paper, we are concerned with the null controllability of a linear population dynamics cascade systems (or the so-called prey-predator models) with two different dispersion coefficients which degenerate in the boundary and with one control force. We develop first a Carleman type inequality for its adjoint system, and then an observability inequality which allows us to deduce the existence of a control acting on a subset of the space domain which steers both populations of a certain age to extinction in a finite time.

## 1. Introduction

We consider the coupled population cascade system

$$\begin{aligned}
 \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - (k_1(x)y_x)_x + \mu_1(t, a, x)y &= \vartheta \chi_\omega & \text{in } Q, \\
 \frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} - (k_2(x)p_x)_x + \mu_2(t, a, x)p + \mu_3(t, a, x)y &= 0 & \text{in } Q, \\
 y(t, a, 1) = y(t, a, 0) = p(t, a, 1) = p(t, a, 0) &= 0 & \text{on } (0, A) \times (0, T), \\
 y(0, a, x) = y_0(a, x); p(0, a, x) = p_0(a, x) & & \text{in } Q_A, \\
 y(t, 0, x) &= \int_0^A \beta_1(t, a, x)y(t, a, x)da & \text{in } Q_T, \\
 p(t, 0, x) &= \int_0^A \beta_2(t, a, x)p(t, a, x)da & \text{in } Q_T,
 \end{aligned} \tag{1.1}$$

where  $Q = (0, T) \times (0, A) \times (0, 1)$ ,  $Q_A = (0, A) \times (0, 1)$ ,  $Q_T = (0, T) \times (0, 1)$  and we will denote  $q = (0, T) \times (0, A) \times \omega$ . The system (1.1) models the dispersion of a gene in two given populations which are in interaction. In this case,  $x$  represents the gene type and  $y(t, a, x)$  and  $p(t, a, x)$  as the distributions of individuals of age  $a$  at time  $t$  and of gene type  $x$  of both populations. The parameters  $\beta_1(t, a, x)$  (respectively  $\beta_2(t, a, x)$ ),  $\mu_1(t, a, x)$

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(respectively  $\mu_2(t, a, x)$ ) are respectively the natural fertility and mortality rates of individuals of age  $a$  at time  $t$  and of gene type  $x$  of the population whose distribution is  $y$  (respectively  $p$ ),  $\mu_3$  can be interpreted as the interaction coefficient between two populations (cancer cells and healthy cells for instance) which depends on  $x$ ,  $t$  and  $a$ , the subset  $\omega$  is the region where a control  $\vartheta$  is acting. Such a control corresponds to an external supply or to removal of individuals on the subdomain  $\omega$ . Finally,  $\int_0^A \beta_1(t, a, x)y(t, a, x)da$  and  $\int_0^A \beta_2(t, a, x)p(t, a, x)da$  are the distributions of the newborns of the two populations that are of gene type  $x$  at time  $t$ .

The control problems of (1.1) or in general of coupled systems take an intense interest and are widely investigated in many papers, among them we find [3], [7], [17] and the references therein. In fact, in [3] the authors studied a coupled reaction-diffusion equations describing interaction between a prey population and predator population. The goal of this work was to look for a suitable control supported on a small spatial subdomain which guarantees the stabilization of the predator population to zero. In [17], the objective was different. More precisely, the authors considered an age-dependent prey-predator system and they proved the existence and uniqueness for an optimal control (called also "optimal effort") which gives the maximal harvest via the study of the optimal harvesting problem associated to their coupled model.

However, the previous results were found in the case when the diffusion coefficients are constants. This leads Ait Ben Hassi et al. in [7] to generalize the model of [3] and investigate a semilinear parabolic cascade systems with two different diffusion coefficients allowed to depend on the space variable and degenerate at the left boundary of the space domain. Moreover, the purpose of this paper was to show the null controllability via a Carleman type inequality of the adjoint problem of the associated linearized system using the results of [8] (or [12]) and with the help of the Schauder fixed point theorem. On the other hand, a massive interest was given to the question of null controllability of the population dynamics models in the case of one equation both in the case without diffusion (see for example [9]) and with diffusion (see for instance [1, 2, 4, 5, 15] in the case of a constant diffusion coefficient). Recently, a more general case was investigated by B. Ainseba and al. in [6] and [13]. Indeed, in [6] the authors allowed the dispersion coefficient to depend on the variable  $x$  and verifies  $k(0) = 0$  (i.e, the coefficient of dispersion  $k$  degenerates at 0) and they tried to obtain the null controllability in such a situation with  $\beta \in L^\infty$  basing on the work done in [8] for the degenerate heat equation to establish a new Carleman estimate for a suitable full adjoint system and afterwards his observability inequality. However, the main controllability result of [6] was shown under the condition  $T \geq A$  (as in [9]) and this constitutes a restrictiveness on the "optimality" of the control time  $T$  since it means, for example, that for a pest population whose the maximal age  $A$  may equal to a many days (may be many months or years) we need much time to bring the population to the zero equilibrium. In the same trend and to overcome the condition  $T \geq A$ , L. Maniar et al in [13] suggested the fixed point technique implemented in [15] and which requires that the fertility rate must belong to  $C^2(Q)$  and consists briefly to demonstrate in a first time the null controllability for an intermediate system with a fertility function  $b \in L^2(Q_T)$  instead of  $\int_0^A \beta(t, a, x)y(t, a, x)da$  and to achieve the task via a Leray-Schauder theorem.

But up now, little is known about the null controllability question of population dynamics

cascade systems both in degenerate and nondegenerate cases to our knowledge and the work done in this paper will address to such a control problem and it will be a generalization of the results established in [6] and [13]. More precisely, following the strategy of [7] we expect in this contribution to prove the null controllability of system (1.1) when  $T \in (0, \delta)$  where  $\delta \in (0, A)$  small enough in the case of one control force. That is, we show that for all  $y_0, p_0 \in L^2(Q_A)$  and  $\delta \in (0, A)$  small enough, there exists a control  $\vartheta \in L^2(q)$  such that the associated solution of (1.1) verifies

$$\begin{cases} y(T, a, x) = 0, & \text{a.e. in } (\delta, A) \times (0, 1), \\ p(T, a, x) = 0, & \text{a.e. in } (\delta, A) \times (0, 1). \end{cases} \quad (1.2)$$

Such a result is gotten under the conditions that all the natural rates possess an  $L^\infty$ -regularity (see (2.4) beneath) and the dispersion coefficients are different and depend on the gene type with a degeneracy in the left hand side of its domain, i.e  $k_i(0) = 0; i = 1, 2$  (e.g  $k_i = x^\alpha$ ,  $\alpha > 0$ ). In this case, we say that (1.1) is a degenerate population dynamics cascade system. Genetically speaking, such a property is natural since it means that if each population is not of a gene type, it can not be transmitted to its offspring.

The remainder of this paper is organized as follows: in Section 2, we give the well-posedness result of system (1.1) and we bring out a Carleman inequality for an intermediate trivial adjoint system which helps us to prove the main Carleman estimate for the full adjoint model. With the aid of this inequality, we establish in Section 3 the observability inequality and show the main result of the null controllability of (1.1). The last section takes the form of an appendix wherein we will give the proofs of some basic tools.

## 2. Well-posedness and Carleman estimates

**2.1. Well-posedness result.** For this section and for the sequel, we assume that the dispersion coefficients  $k_i, i = 1, 2$  satisfy the hypotheses

$$\begin{cases} k_i \in C([0, 1]) \cap C^1((0, 1]), \quad k_i > 0 \text{ in } (0, 1] \text{ and } k_i(0) = 0, \\ \exists \gamma \in [0, 1] : x k_i'(x) \leq \gamma k_i(x), \quad x \in [0, 1]. \end{cases} \quad (2.3)$$

The last hypothesis on  $k_i$  means in the case of  $k(x) = x^{\alpha_i}$  that  $0 \leq \alpha_i < 1$ . Similarly, all results of this paper can be obtained also in the case of  $1 \leq \alpha_i < 2$  taking, instead of Dirichlet condition on  $x = 0$ , the Neumann condition  $(k_i(x)u_x)(0) = 0$ . On the other hand, we assume that the rates  $\mu_1, \mu_2, \mu_3, \beta_1$  and  $\beta_2$  verify

$$\begin{cases} \mu_1, \mu_2, \mu_3, \beta_1, \beta_2 \in L^\infty(Q), \quad \mu_1, \mu_2, \mu_3, \beta_1, \beta_2 \geq 0 \text{ a.e in } Q, \\ \beta_i(., 0, .) \equiv 0 \text{ a.e. in } (0, T) \times (0, 1), \quad \text{for } i = 1, 2. \end{cases} \quad (2.4)$$

The third assumption in (2.4) on the fertility rates  $\beta_1$  and  $\beta_2$  is natural since the newborns are not fertile.

As in [13], we discuss the well-posedness of (1.1) by introducing the weighted spaces  $H_{k_i}^1(0, 1)$  and  $H_{k_i}^2(0, 1)$  defined by

$$\begin{cases} H_{k_i}^1(0, 1) := \{u \in L^2(0, 1) : u \text{ is abs. cont. in } [0, 1] : \sqrt{k_i}u_x \in L^2(0, 1), u(1) = u(0) = 0\}, \\ H_{k_i}^2(0, 1) := \left\{u \in H_{k_i}^1(0, 1) : k_i(x)u_x \in H^1(0, 1)\right\}, \end{cases}$$

endowed respectively with the norms

$$\begin{cases} \|u\|_{H_{k_i}^1(0,1)}^2 := \|u\|_{L^2(0,1)}^2 + \|\sqrt{k_i}u_x\|_{L^2(0,1)}^2, & u \in H_{k_i}^1(0,1), \\ \|u\|_{H_{k_i}^2(0,1)}^2 := \|u\|_{H_{k_i}^1(0,1)}^2 + \|(k_i(x)u_x)_x\|_{L^2(0,1)}^2, & u \in H_{k_i}^2(0,1), \end{cases}$$

with  $i = 1, 2$  (see [7], [8], [12] or the references therein for the properties of such a spaces). We recall from [11, 12] that the operators  $C_i u := (k_i(x)u_x)_x$ ,  $u \in D(C_i) = H_{k_i}^2(0,1)$ ,  $i = 1, 2$  are closed self-adjoint and negative with dense domains in  $L^2(0,1)$ .

On the other hand, in the Hilbert space  $\mathbb{H} = (L^2((0,A) \times (0,1)))^2$ , the system (1.1) can be rewritten abstractly as an inhomogeneous Cauchy problem in the following way

$$\begin{cases} X'(t) = \mathbb{A}X(t) + B(t)X(t) + f(t), \\ X(0) = \begin{pmatrix} y_0 \\ p_0 \end{pmatrix}, \end{cases} \quad (2.5)$$

where  $X(t) = \begin{pmatrix} y(t) \\ p(t) \end{pmatrix}$ ,  $\mathbb{A} = \begin{pmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_2 \end{pmatrix}$ ;  $D(\mathbb{A}) = D(\mathcal{A}_1) \times D(\mathcal{A}_2)$ ,

$f(t) = \begin{pmatrix} \vartheta(t, \cdot, \cdot) \chi_\omega(\cdot) \\ 0 \end{pmatrix}$ ,  $B(t) = \begin{pmatrix} M_{\mu_1(t)} & 0 \\ M_{\mu_3(t)} & M_{\mu_2(t)} \end{pmatrix}$ , where  $M_{\mu_j(t)} w = -\mu_j(t)w$ , the operators  $\mathcal{A}_1 : L^2((0,A) \times (0,1)) \rightarrow L^2((0,A) \times (0,1))$  and  $\mathcal{A}_2 : L^2((0,A) \times (0,1)) \rightarrow L^2((0,A) \times (0,1))$  are defined respectively by:

$$\begin{cases} \mathcal{A}_1 \theta(a, x) = -\frac{\partial \theta}{\partial a} + (k_1(x)\theta_x)_x, \quad \forall \theta \in D(\mathcal{A}_1), \\ D(\mathcal{A}_1) = \{\theta(a, x) : \theta, \mathcal{A}_1 \theta \in L^2((0,A) \times (0,1)), \theta(a, 0) = \theta(a, 1) = 0, \theta(0, x) = \int_0^A \beta_1(a, x)\theta(a, x)da\}, \end{cases} \quad (2.6)$$

and

$$\begin{cases} \mathcal{A}_2 \theta(a, x) = -\frac{\partial \theta}{\partial a} + (k_2(x)\theta_x)_x, \quad \forall \theta \in D(\mathcal{A}_2), \\ D(\mathcal{A}_2) = \{\theta(a, x) : \theta, \mathcal{A}_2 \theta \in L^2((0,A) \times (0,1)), \theta(a, 0) = \theta(a, 1) = 0, \theta(0, x) = \int_0^A \beta_2(a, x)\theta(a, x)da\}. \end{cases} \quad (2.7)$$

It is well-known, from [16] and the references therein that the operators  $\mathcal{A}_1$  and  $\mathcal{A}_2$  defined above generate a  $C_0$ -semigroups. On the other hand, one can see that the operator  $\mathbb{A}$  is diagonal and  $B(t)$  is a bounded perturbation. Therefore, the following well-posedness result holds (see for instance [7] for a similar result of cascade parabolic equations).

**Theorem 2.1.** *i) The operator  $\mathbb{A}$  generates a  $C_0$ -semigroup.*

*ii) Under the assumptions (2.3) and (2.4) and for all  $\vartheta \in L^2(Q)$  and  $(y_0, p_0) \in (L^2(Q_A))^2$ , the system (1.1) admits a unique solution  $(y, p)$ . This solution belongs to  $E := C([0, T], (L^2((0,A) \times (0,1)))^2) \cap C([0, A], (L^2((0,T) \times (0,1)))^2) \cap L^2((0,T) \times (0,A), H_{k_1}^1(0,1) \times H_{k_2}^1(0,1))$ . Moreover, the solution of (1.1) satisfies the following inequality*

$$\begin{aligned} & \sup_{t \in [0, T]} \|(y(t), p(t))\|_{L^2(Q_A) \times L^2(Q_A)}^2 + \sup_{a \in [0, A]} \|(y(a), p(a))\|_{L^2(Q_T) \times L^2(Q_T)}^2 \\ & + \int_0^1 \int_0^A \int_0^T ((\sqrt{k_1}y_x)^2 + (\sqrt{k_2}p_x)^2) dt da dx \\ & \leq C \left( \int_q \vartheta^2 dt da dx + \|(y_0, p_0)\|_{L^2(Q_A) \times L^2(Q_A)}^2 \right). \end{aligned} \quad (2.8)$$

**2.2. Carleman inequality results.** In this paragraph, we show a Carleman type inequality for the following adjoint system of (1.1)

$$\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k_1(x)u_x)_x - \mu_1(t, a, x)u - \mu_3(t, a, x)v &= -\beta_1(t, a, x)u(t, 0, x) && \text{in } Q, \\
\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + (k_2(x)v_x)_x - \mu_2(t, a, x)v &= -\beta_2(t, a, x)v(t, 0, x) && \text{in } Q, \\
u(t, a, 1) = u(t, a, 0) = v(t, a, 1) = v(t, a, 0) &= 0 && \text{on } (0, T) \times (0, A), \\
u(T, a, x) &= u_T(a, x) && \text{in } Q_A, \\
v(T, a, x) &= v_T(a, x) && \text{in } Q_A, \\
u(t, A, x) = v(t, A, x) &= 0 && \text{in } Q_T.
\end{aligned} \tag{2.9}$$

To do this, we prove firstly the Carleman estimate for the following intermediate system

$$\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k_1(x)u_x)_x - \mu_1(t, a, x)u - \mu_3(t, a, x)v &= h_1 && \text{in } Q, \\
\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + (k_2(x)v_x)_x - \mu_2(t, a, x)v &= h_2 && \text{in } Q, \\
u(t, a, 1) = u(t, a, 0) = v(t, a, 1) = v(t, a, 0) &= 0 && \text{on } (0, T) \times (0, A), \\
u(T, a, x) &= u_T(a, x) && \text{in } Q_A, \\
v(T, a, x) &= v_T(a, x) && \text{in } Q_A, \\
u(t, A, x) = v(t, A, x) &= 0 && \text{in } Q_T,
\end{aligned} \tag{2.10}$$

with  $(u_T, v_T) \in (L^2(Q_A))^2$  and  $h_1, h_2 \in L^2(Q)$ . Such a system can be rewritten in the following way

$$\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k_1(x)u_x)_x - \mu_1(t, a, x)u &= h_1 + \mu_3(t, a, x)v && \text{in } Q, \\
u(t, a, 1) = u(t, a, 0) &= 0 && \text{on } (0, T) \times (0, A), \\
u(T, a, x) &= u_T(a, x) && \text{in } Q_A, \\
u(t, A, x) &= 0 && \text{in } Q_T,
\end{aligned} \tag{2.11}$$

where  $v$  is the solution of

$$\begin{aligned}
\frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + (k_2(x)u_x)_x - \mu_2(t, a, x)v &= h_2 && \text{in } Q, \\
v(t, a, 1) = v(t, a, 0) &= 0 && \text{on } (0, T) \times (0, A), \\
v(T, a, x) &= v_T(a, x) && \text{in } Q_A, \\
v(t, A, x) &= 0 && \text{in } Q_T.
\end{aligned} \tag{2.12}$$

Classically, the proof of such a kind of estimates is based tightly on the choice of the so-called weight functions. In our case, these functions are set in the following way

$$\left\{ \begin{aligned} &\varphi_i(t, a, x) := \Theta(t, a)\psi_i(x), i = 1, 2, \\ &\Theta(t, a) := \frac{1}{(t(T-t))^4 a^4}, \\ &\psi_i(x) := \lambda_i \left( \int_0^x \frac{r}{k_i(r)} dr - d_i \right), \\ &\phi(t, a, x) = \Theta(a, t)e^{\kappa\sigma(x)}, \Phi(t, a, x) = \Theta(a, t)\Psi(x), \Psi(x) = e^{\kappa\sigma(x)} - e^{2\kappa\|\sigma\|_\infty}, \end{aligned} \right. \tag{2.13}$$

where  $\sigma$  is the function given by

$$\begin{cases} \sigma \in C^2([0, 1]), \sigma(x) > 0 \text{ in } (0, 1), \sigma(0) = \sigma(1) = 0, \\ \sigma_x(x) \neq 0 \text{ in } [0, 1] \setminus \omega_0, \end{cases} \quad (2.14)$$

$\omega_0 \subseteq \omega$  is an open subset. The existence of this function is proved in [14, Lemma 1.1].  $\lambda_i$ ,  $d_i$  for  $i = 1, 2$  and  $\kappa$  are supposed to verify following assumptions

$$\begin{cases} d_1 > \frac{1}{k_1(1)(2-\gamma)}, \frac{\lambda_1}{\lambda_2} \geq \frac{d_2}{d_1 - \int_0^1 \frac{r}{k_1(r)} dr}, \\ \kappa \geq \frac{4 \ln 2}{\|\sigma\|_\infty}, d_2 \geq \frac{5}{k_2(1)(2-\gamma)}, \end{cases} \quad (2.15)$$

with  $\lambda_2 \in I = [\frac{k_2(1)(2-\gamma)(e^{2\kappa\|\sigma\|_\infty} - 1)}{d_2 k_2(1)(2-\gamma) - 1}, \frac{4(e^{2\kappa\|\sigma\|_\infty} - e^{\kappa\|\sigma\|_\infty})}{3d_2}]$  which can be shown not empty (see Lemma 4.3 in the appendix). On other hand, in the light of the first and the fourth conditions in (2.15) on  $d_1$  and  $d_2$ , one can observe that  $\psi_i(x) < 0$  for all  $x \in [0, 1]$ , and  $\Theta(t, a) \rightarrow +\infty$  as  $t \rightarrow 0^+, T^-$  and  $a \rightarrow 0^+$ .

Now, we state the first result of this section which is the intermediate Carleman estimate satisfied by solution of system (2.10).

**Theorem 2.2.** *Assume that  $k_i$  satisfy the hypotheses (2.3) and let  $A > 0$  and  $T > 0$  be given. Then, there exist two positive constants  $C$  and  $s_0$ , such that every solution  $(u, v)$  of (2.10) satisfies, for all  $s \geq s_0$ , the following inequality*

$$\begin{aligned} & \int_Q \left( s^3 \Theta^3 \frac{x^2}{k_1(x)} u^2 + s \Theta k_1(x) u_x^2 \right) e^{2s\varphi_1} dt da dx + \int_Q \left( s^3 \Theta^3 \frac{x^2}{k_2(x)} v^2 + s \Theta k_2(x) v_x^2 \right) e^{2s\varphi_2} dt da dx \\ & \leq C \left( \int_Q (h_1^2 + h_2^2) e^{2s\Phi} dt da dx + \int_Q s^3 \Theta^3 (u^2 + v^2) e^{2s\Phi} dt da dx \right). \end{aligned} \quad (2.16)$$

The proof of Theorem 2.2 needs two basic results. These results are concerned with Carleman type inequalities in both cases degenerate and nondegenerate. The first one is stated in the following proposition

**Proposition 2.3.** *Consider the following system with  $h \in L^2(Q)$ ,  $\mu \in L^\infty(Q)$  and  $k$  verifies the hypotheses (2.3)*

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k(x)u_x)_x - \mu(t, a, x)u &= h, \\ u(t, a, 1) = u(t, a, 0) &= 0, \\ u(T, a, x) &= u_T(a, x), \\ u(t, A, x) &= 0. \end{aligned} \quad (2.17)$$

Then, there exist two positive constants  $C$  and  $s_0$ , such that every solution of (2.17) satisfies, for all  $s \geq s_0$ , the following inequality

$$\begin{aligned} & s^3 \int_Q \Theta^3 \frac{x^2}{k(x)} u^2 e^{2s\varphi} dt da dx + s \int_Q \Theta k(x) u_x^2 e^{2s\varphi} dt da dx \\ & \leq C \left( \int_Q |h|^2 e^{2s\varphi} dt da dx + sk(1) \int_0^A \int_0^T \Theta u_x^2(a, t, 1) e^{2s\varphi(a, t, 1)} dt da \right), \end{aligned} \quad (2.18)$$

where  $\varphi$  and  $\Theta$  are the weight functions defined by

$$\begin{cases} \varphi(t, a, x) := \Theta(t, a)\psi(x) \text{ with :} \\ \Theta(t, a) := \frac{1}{(t(T-t))^4 a^4}, \\ \psi(x) := c_1 \left( \int_0^x \frac{r}{k(r)} dr - c_2 \right). \end{cases} \quad (2.19)$$

with  $c_2 > \frac{1}{k(1)(2-\gamma)}$ ,  $c_1 > 0$  and  $\gamma$  is the parameter defined by (2.3).

For the proof of this proposition, we refer the reader to [13, Proposition 3.1]. The second result is the following

**Proposition 2.4.** *Let us consider the following system*

$$\begin{aligned} \frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} + (k(x)z_x)_x - c(t, a, x)z &= h \quad \text{in } Q_b, \\ z(t, a, b_1) = z(t, a, b_2) &= 0 \quad \text{on } (0, T) \times (0, A), \end{aligned} \quad (2.20)$$

where  $Q_b := (0, T) \times (0, A) \times (b_1, b_2)$ ,  $(b_1, b_2) \subset [0, 1]$ ,  $h \in L^2(Q_b)$ ,  $k \in C^1([0, 1])$  is a strictly positive function and  $c \in L^\infty(Q_b)$ . Then, there exist two positive constants  $C$  and  $s_0$ , such that for any  $s \geq s_0$ ,  $z$  verifies the following estimate

$$\begin{aligned} & \int_{Q_b} (s^3 \phi^3 z^2 + s \phi z_x^2) e^{2s\Phi} dt da dx \\ & \leq C \left( \int_{Q_b} h^2 e^{2s\Phi} dt da dx + \int_\omega \int_0^A \int_0^T s^3 \phi^3 z^2 e^{2s\Phi} dt da dx \right), \end{aligned} \quad (2.21)$$

where  $\phi$ ,  $\Theta$  and  $\Phi$  are defined by (2.13) and  $\sigma$  by (2.14).

For the proof of Proposition 2.4, a careful computations allow us to adapt the same procedure of [2, Lemma 2.1] to show (2.21) in case where  $k$  is a positive general nondegenerate coefficient, with our weight function  $\Theta(t, a) = \frac{1}{t^4(T-t)^4 a^4}$  and the source term  $h$ .

Besides the two Propositions 2.3 and 2.4, we must bring out another important result

**Lemma 2.5.** *Under assumptions (2.15), the functions  $\varphi_1$ ,  $\varphi_2$  and  $\Phi$  defined by (2.13) satisfy the following inequalities*

$$\begin{cases} \varphi_1 \leq \varphi_2, \\ \frac{4}{3}\Phi < \varphi_2 \leq \Phi. \end{cases} \quad (2.22)$$

*Proof.* By the definitions of  $\varphi_1$ ,  $\varphi_2$  and  $\Phi$  and taking into account that  $\Theta$  is positive, showing the results of (2.22) is equivalent to show

$$\begin{cases} \psi_1 \leq \psi_2, \\ \frac{4}{3}\Psi < \psi_2 \leq \Psi. \end{cases} \quad (2.23)$$

The first inequality in (2.23) is assured by the second assumption in (2.15) while the second one is deduced from  $\lambda_2 \in I = [\frac{k_2(1)(2-\gamma)(e^{2\kappa\|\sigma\|_\infty}-1)}{d_2 k_2(1)(2-\gamma)-1}, \frac{4(e^{2\kappa\|\sigma\|_\infty}-e^{\kappa\|\sigma\|_\infty})}{3d_2}]$  and this achieves the proof.  $\square$

Now, we can address the proof of Theorem 2.2.



*Proof.* Let us introduce the smooth cut-off function  $\xi : \mathbb{R} \rightarrow \mathbb{R}$  defined as follows

$$\begin{cases} 0 \leq \xi(x) \leq 1, & x \in \mathbb{R}, \\ \xi(x) = 1, & x \in [0, \frac{2x_1+x_2}{3}], \\ \xi(x) = 0, & x \in [\frac{x_1+2x_2}{3}, 1]. \end{cases} \quad (2.24)$$

Let  $u$  and  $v$  be respectively the solutions of (3.75) and (3.76). Set  $w := \xi u$ ,  $z := \xi v$  and put  $\omega' = (\frac{2x_1+x_2}{3}, \frac{x_1+2x_2}{3})$ . Then,  $(w, z)$  satisfies the following system

$$\begin{aligned} \frac{\partial w}{\partial t} + \frac{\partial w}{\partial a} + (k_1(x)w_x)_x - \mu_1(t, a, x)w &= \mu_3(t, a, x)z + \xi h_1 + (k_1\xi_x u)_x + \xi_x k_1 u_x & \text{in } Q, \\ \frac{\partial z}{\partial t} + \frac{\partial z}{\partial a} + (k_2(x)z_x)_x - \mu_2(t, a, x)z &= \xi h_2 + (k_2\xi_x v)_x + \xi_x k_2 v_x & \text{in } Q, \\ w(t, a, 1) = w(t, a, 0) = z(t, a, 1) = z(t, a, 0) &= 0 & \text{on } (0, T) \times (0, A), \\ w(T, a, x) = w_T(a, x) & & \text{in } Q_A, \\ z(T, a, x) = z_T(a, x) & & \text{in } Q_A, \\ w(t, A, x) = z(t, A, x) = 0 & & \text{in } Q_T. \end{aligned} \quad (2.25)$$

Using Proposition 2.3 for the inhomogeneous term  $\xi(h_1 + \mu_3 v) + (k_1 \xi_x u)_x + \xi_x k_1 u_x$ , the definition of  $\xi$  and Young inequality, we get the following inequality

$$\begin{aligned} & \int_Q (s\Theta k_1 w_x^2 + s^3 \Theta^3 \frac{x^2}{k_1} w^2) e^{2s\varphi_1} dt da dx \\ & \leq C \left( \int_Q [\xi^2 (h_1 + \mu_3 v)^2 + ((k_1 \xi_x u)_x + \xi_x k_1 u_x)^2] e^{2s\varphi_1} dt da dx \right. \\ & \quad \left. + s k_1(1) \int_0^A \int_0^T \Theta w_x^2(t, a, 1) e^{2s\varphi_1(t, a, 1)} dt da \right) \\ & \leq \overline{C} \int_Q [\mu_3^2 z^2 + \xi^2 h_1^2 + ((k_1 \xi_x u)_x + \xi_x k_1 u_x)^2] e^{2s\varphi_1} dt da dx. \end{aligned} \quad (2.26)$$

Thanks again to the definition of  $\xi$ , we have

$$\begin{aligned} & \int_0^1 ((k_1 \xi_x u)_x + \xi_x k_1 u_x)^2 e^{2s\varphi_1} dx \\ & \leq \int_{\omega'} (8(k_1 \xi_x)^2 u_x^2 + 2((k_1 \xi_x)_x)^2 u^2) e^{2s\varphi_1} dx \\ & \leq C \int_{\omega'} (u^2 + u_x^2) e^{2s\varphi_1} dx. \end{aligned} \quad (2.27)$$

On the other hand, since  $\frac{x^2}{k_2(x)}$  is non-decreasing, with the help of Hardy-Poincaré inequality stated in [8] and since  $\varphi_1 \leq \varphi_2$  we get

$$\int_0^1 \mu_3^2 z^2 e^{2s\varphi_1} dx \leq \frac{\|\mu_3\|_\infty^2}{k_2(1)} \int_0^1 \frac{k_2(x)}{x^2} (ze^{s\varphi_2})^2 dx \leq C \frac{\|\mu_3\|_\infty^2}{k_2(1)} \int_0^1 k_2(x) ((ze^{s\varphi_2})_x)^2 dx.$$

Thus, from the definition of  $\psi_2$ , we obtain

$$\int_0^1 \mu_3^2 z^2 e^{2s\varphi_1} dx \leq C \int_0^1 k_2(x) z_x^2 e^{2s\varphi_2} dx + C \int_0^1 s^2 \Theta^2 \frac{x^2}{k_2(x)} z^2 e^{2s\varphi_2} dx.$$



Hence, for  $s$  quite large we get

$$\int_0^1 \mu_3^2 z^2 e^{2s\varphi_1} dx \leq \frac{1}{2} \int_0^1 s\Theta k_2(x) z_x^2 e^{2s\varphi_2} dx + \frac{1}{2} \int_0^1 s^3 \Theta^3 \frac{x^2}{k_2(x)} z^2 e^{2s\varphi_2} dx. \quad (2.28)$$

Combining (2.26), (2.27) and (2.28), for  $s$  quite large the following inequality holds

$$\begin{aligned} & \int_Q (s\Theta k_1 w_x^2 + s^3 \Theta^3 \frac{x^2}{k_1} w^2) e^{2s\varphi_1} dt d\alpha dx \\ & \leq \overline{C} \int_Q h_1^2 e^{2s\varphi_1} dt d\alpha dx + \frac{1}{2} \int_Q (s\Theta k_2(x) z_x^2 + s^3 \Theta^3 \frac{x^2}{k_2(x)} z^2) e^{2s\varphi_2} dt d\alpha dx \\ & + C_1 \int_{\omega'} \int_0^A \int_0^T (u^2 + u_x^2) e^{2s\varphi_1} dt d\alpha dx. \end{aligned} \quad (2.29)$$

Applying the same way with  $\xi h_2 + (k_2 \xi_x v)_x + \xi_x k_2 v_x$  we obtain

$$\begin{aligned} & \int_Q (s\Theta k_2 z_x^2 + s^3 \Theta^3 \frac{x^2}{k_2} z^2) e^{2s\varphi_2} dt d\alpha dx \\ & \leq C_2 \int_Q h_2^2 e^{2s\varphi_2} dt d\alpha dx + C_3 \int_{\omega'} \int_0^A \int_0^T (v^2 + v_x^2) e^{2s\varphi_2} dt d\alpha dx. \end{aligned} \quad (2.30)$$

Therefore, for  $s$  quite large we conclude by inequalities (2.29) and (2.30) and again  $\varphi_1 \leq \varphi_2$  that

$$\begin{aligned} & \int_Q (s\Theta k_1 w_x^2 + s^3 \Theta^3 \frac{x^2}{k_1} w^2) e^{2s\varphi_1} dt d\alpha dx + \int_Q (s\Theta k_2 z_x^2 + s^3 \Theta^3 \frac{x^2}{k_2} z^2) e^{2s\varphi_2} dt d\alpha dx \\ & \leq C_4 \int_Q (h_1^2 + h_2^2) e^{2s\varphi_2} dt d\alpha dx + C_5 \int_{\omega'} \int_0^A \int_0^T (u^2 + v^2 + u_x^2 + v_x^2) e^{2s\varphi_2} dt d\alpha dx. \end{aligned}$$

Using Caccioppoli's inequality (4.87), the last inequality becomes

$$\begin{aligned} & \int_Q (s\Theta k_1 w_x^2 + s^3 \Theta^3 \frac{x^2}{k_1} w^2) e^{2s\varphi_1} dt d\alpha dx + \int_Q (s\Theta k_2 z_x^2 + s^3 \Theta^3 \frac{x^2}{k_2} z^2) e^{2s\varphi_2} dt d\alpha dx \\ & \leq C_6 \int_Q (h_1^2 + h_2^2) e^{2s\varphi_2} dt d\alpha dx + C_7 \int_Q s^2 \Theta^2 (u^2 + v^2) e^{2s\varphi_2} dt d\alpha dx. \end{aligned} \quad (2.31)$$

Now, let  $W := \eta u$  and  $Z := \eta v$  with  $\eta = 1 - \xi$ . Then  $W$  and  $Z$  are supported in  $(x_1, 1)$  and verify the following system

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial a} + (k_1(x) W_x)_x - \mu_1(t, a, x) W = \mu_3(t, a, x) Z + \eta h_1 + (k_1 \eta_x u)_x + \eta_x k_1 u_x \quad \text{in } Q_{x_1}, \quad (2.32)$$

$$\frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial a} + (k_2(x) Z_x)_x - \mu_2(t, a, x) Z = \eta h_2 + (k_2 \eta_x v)_x + \eta_x k_2 v_x \quad \text{in } Q_{x_1},$$

$$W(t, a, 1) = W(t, a, x_1) = Z(t, a, 1) = Z(t, a, x_1) = 0 \quad \text{on } (0, T) \times (0, A),$$

$$W(t, a, x) = W_T(a, x) \quad \text{in } Q_A,$$

$$Z(t, a, x) = Z_T(a, x) \quad \text{in } Q_A,$$

$$W(t, A, x) = Z(t, A, x) = 0 \quad \text{in } Q_T,$$

where,  $Q_{x_1} := (0, T) \times (0, A) \times (x_1, 1)$ . Then, the system satisfied by  $W$  and  $Z$  is non-degenerate. Hence, applying Proposition 2.4 on the first equation of (2.32) for  $b_1 = x_1$ ,

$b_2 = 1$  and  $h := \eta(h_1 + \mu_3 v) + (k_1 \eta_x u)_x + \eta_x k_1 u_x$ , with the aid of Caccioppoli's inequality stated in [13, Lemma 5.1], thanks to the definition of  $\eta$  and Young inequality and taking  $s$  quite large we obtain the following estimate

$$\begin{aligned}
& \int_Q (s^3 \phi^3 W^2 + s \phi W_x^2) e^{2s\Phi} dt dadx \\
& \leq C \left( \int_Q (\eta(h_1 + \mu_3 v) + (k\eta_x u)_x + k\eta_x u_x)^2 e^{2s\Phi} dt dadx + \int_\omega \int_0^A \int_0^T s^3 \Theta^3 u^2 e^{2s\Phi} dt dadx \right) \\
& \leq \tilde{C} \left( \int_Q \eta^2 (h_1 + \mu_3 v)^2 e^{2s\Phi} + ((k\eta_x u)_x + k\eta_x u_x)^2 e^{2s\Phi} dt dadx + \int_\omega \int_0^A \int_0^T s^3 \Theta^3 u^2 e^{2s\Phi} dt dadx \right) \\
& \leq \tilde{C} \left( \int_Q \eta^2 (h_1 + \mu_3 v)^2 e^{2s\Phi} dt dadx + \int_{\omega'} \int_0^A \int_0^T (8(k\eta_x)^2 u_x^2 + 2((k\eta_x)_x)^2 u^2) e^{2s\Phi} dt dadx \right. \\
& \quad \left. + \int_\omega \int_0^A \int_0^T s^3 \Theta^3 u^2 e^{2s\Phi} dt dadx \right) \\
& \leq \tilde{C}_1 \left( \int_Q \eta^2 (h_1 + \mu_3 v)^2 e^{2s\Phi} dt dadx + \int_{\omega'} \int_0^A \int_0^T (u_x^2 + u^2) e^{2s\Phi} dt dadx + \int_\omega \int_0^A \int_0^T s^3 \Theta^3 u^2 e^{2s\Phi} dt dadx \right) \\
& \leq \tilde{C}_2 \left( \int_Q \eta^2 (h_1 + \mu_3 v)^2 e^{2s\Phi} dt dadx + \int_\omega \int_0^A \int_0^T s^3 \Theta^3 u^2 e^{2s\Phi} dt dadx \right) \\
& \leq \tilde{C}_3 \left( \int_Q (h_1^2 + \mu_3^2 Z^2) e^{2s\Phi} dt dadx + \int_q s^3 \Theta^3 u^2 e^{2s\Phi} dt dadx \right), \tag{2.33}
\end{aligned}$$

with  $\Phi$  and  $\phi$  are defined in (2.13) and  $\omega'$  is defined in the beginning of the proof. On the other hand, using the fact that  $x \mapsto \frac{x^2}{k_2(x)}$  is non-decreasing, Hardy-Poincaré inequality for the function  $Ze^{s\Phi}$  and the definition of  $\psi_2$  we have for  $s$  quite large the following inequality

$$\begin{aligned}
& \int_Q \mu_3^2 Z^2 e^{2s\Phi} dx \leq c \left( \int_Q k_2(x) Z_x^2 e^{2s\Phi} dt dadx + \int_Q s^2 \Theta^2 \frac{x^2}{k_2(x)} Z^2 e^{2s\Phi} dt dadx \right) \\
& \leq \frac{1}{2} \int_Q (s^3 \phi^3 Z^2 + s \phi Z_x^2) e^{2s\Phi} dt dadx. \tag{2.34}
\end{aligned}$$

Therefore, injecting (2.34) in (2.33) we get

$$\begin{aligned}
& \int_Q (s^3 \phi^3 W^2 + s \phi W_x^2) e^{2s\Phi} dt dadx \tag{2.35} \\
& \leq C \left( \int_Q h_1^2 e^{2s\Phi} dt dadx + \int_q s^3 \Theta^3 u^2 e^{2s\Phi} dt dadx \right) + \frac{1}{2} \int_Q (s^3 \phi^3 Z^2 + s \phi Z_x^2) e^{2s\Phi} dt dadx.
\end{aligned}$$

Replying the same argument for the source term  $h := \eta h_2 + (k_2 \eta_x v)_x + \eta_x k_2 v_x$  we infer that

$$\int_Q (s^3 \phi^3 Z^2 + s \phi Z_x^2) e^{2s\Phi} dt dadx \leq C_8 \left( \int_Q h_2^2 e^{2s\Phi} dt dadx + \int_q s^3 \Theta^3 v^2 e^{2s\Phi} dt dadx \right). \tag{2.36}$$

Subsequently, combining (2.35) and (2.36) we arrive to

$$\begin{aligned} & \int_Q [s^3 \phi^3(W^2 + Z^2) + s\phi(W_x^2 + Z_x^2)] e^{2s\Phi} dt dx \\ & \leq C_9 \left( \int_Q (h_1^2 + h_2^2) e^{2s\Phi} dt dx + \int_q s^3 \Theta^3(u^2 + v^2) e^{2s\Phi} dt dx \right). \end{aligned} \quad (2.37)$$

Using the fact that  $u = w + W$  and  $v = z + Z$ ,  $\varphi_1 \leq \varphi_2 \leq \Phi$ , the estimates (2.31) and (2.37) lead to estimate (2.16).  $\square$

Using the Theorem 2.2 for a special functions  $h_1$  and  $h_2$ , we are ready to deduce the following result

**Theorem 2.6.** *Assume that the assumptions (2.3) and (2.4) hold. Let  $A > 0$  and  $T > 0$  be given such that  $T \in (0, \delta)$  with  $\delta \in (0, A)$  small enough. Then, there exist positive constants  $C$  (independent of  $\delta$ ) and  $s_0$  such that for all  $s \geq s_0$ , every solution  $(u, v)$  of (2.9) satisfies*

$$\begin{aligned} & \int_Q \left( s^3 \Theta^3 \frac{x^2}{k_1(x)} u^2 + s \Theta k_1(x) u_x^2 \right) e^{2s\varphi_1} dt dx + \int_Q \left( s^3 \Theta^3 \frac{x^2}{k_2(x)} v^2 + s \Theta k_2(x) v_x^2 \right) e^{2s\varphi_2} dt dx \\ & \leq C \left( \int_q s^3 \Theta^3 (u^2 + v^2) e^{2s\Phi} dt dx + \int_0^1 \int_0^\delta (u_T^2(a, x) + v_T^2(a, x)) da dx \right). \end{aligned} \quad (2.38)$$

*Proof.* Let  $h_1 := -\beta_1(t, a, x)u(t, 0, x)$  and  $h_2 := -\beta_2(t, a, x)v(t, 0, x)$ .

Therefore, thanks to (2.16) and (2.4) we have the existence of two positive constants  $C$  and  $s_0$  such that, for all  $s \geq s_0$ , the following inequality holds

$$\begin{aligned} & s^3 \int_Q \Theta^3 \left( \frac{x^2}{k_1(x)} u^2 e^{2s\varphi_1} + \frac{x^2}{k_2(x)} v^2 e^{2s\varphi_2} \right) dt dx + s \int_Q \Theta (k_1(x) u_x^2 e^{2s\varphi_1} + k_2(x) v_x^2 e^{2s\varphi_2}) dt dx \\ & \leq C \left( \int_Q ((\beta_1)^2 u^2(t, 0, x) + (\beta_2)^2 v^2(t, 0, x)) e^{2s\Phi} dt dx + \int_q s^3 \Theta^3 (u^2 + v^2) e^{2s\Phi} dt dx \right) \\ & \leq \tilde{C}_1 \left( \int_0^1 \int_0^T (u^2(t, 0, x) + v^2(t, 0, x)) dt dx + \int_q s^3 \Theta^3 (u^2 + v^2) e^{2s\Phi} dt dx \right) \end{aligned} \quad (2.39)$$

Set  $U(t, a, x) = u(T - t, A - a, x)$  and  $V(t, a, x) = v(T - t, A - a, x)$ . Then, one has

$$\begin{aligned} & \frac{\partial U}{\partial t} + \frac{\partial U}{\partial a} - (k_1(x) U_x)_x + \mu_1(T - t, A - a, x)U + \mu_3(T - t, A - a, x)V = \beta_1(T - t, A - a, x)U(t, A, x), \\ & U(t, a, 1) = U(t, a, 0) = 0, \end{aligned} \quad (2.40)$$

$$U(0, a, x) = U_0(a, x) = u_T(A - a, x),$$

$$U(t, 0, x) = 0,$$

where  $V$  is the solution of

$$\begin{aligned} & \frac{\partial V}{\partial t} + \frac{\partial V}{\partial a} - (k_2(x) V_x)_x + \mu_2(T - t, A - a, x)V = \beta_2(T - t, A - a, x)V(t, A, x), \\ & V(t, a, 1) = V(t, a, 0) = 0, \end{aligned} \quad (2.41)$$

$$V(0, a, x) = V_0(a, x) = v_T(A - a, x),$$

$$V(t, 0, x) = 0.$$

Integrating along the characteristic lines, we get respectively the implicit formulas for the solutions  $U$  of (2.40) and  $V$  of (2.41) given by

$$\begin{cases} U(t, a, \cdot) = \int_0^a S(a-l)(\beta_1(T-t, A-l, \cdot)U(t, A, \cdot) - \mu_3(T-t, A-l, \cdot)V(t, l, \cdot))dl, \\ \text{if } t > a \\ U(t, a, \cdot) = S(t)U_0(a-t, \cdot) + \int_0^t S(t-l)(\beta_1(T-l, A-a, \cdot)U(l, A, \cdot) - \mu_3(T-l, A-a, \cdot)V(l, a, \cdot))dl, \\ \text{if } t \leq a, \end{cases} \quad (2.42)$$

and

$$\begin{cases} V(t, a, \cdot) = \int_0^a L(a-l)\beta_2(T-t, A-l, \cdot)V(t, A, \cdot)dl, & \text{if } t > a \\ V(t, a, \cdot) = L(t)V_0(a-t, \cdot) + \int_0^t L(t-l)\beta_2(T-l, A-a, \cdot)V(l, A, \cdot)dl, & \text{if } t \leq a, \end{cases} \quad (2.43)$$

where  $(S(t))_{t \geq 0}$  and  $(L(t))_{t \geq 0}$  are the bounded semigroups generated respectively by the operators  $A_4 U = -(k_1 U_x)_x + \mu_1(T-t, A-a, x)U$  and  $A_7 V = -(k_2 V_x)_x + \mu_2(T-t, A-a, x)V$ .

Hence, after a careful computations, (2.42) and (2.43) become respectively

$$\begin{cases} u(t, a, \cdot) = \int_0^{A-a} S(A-a-l)(\beta_1(t, A-l, \cdot)u(t, 0, \cdot) - \mu_3(t, A-l, \cdot)v(t, A-l, \cdot))dl, \\ \text{if } a > t + (A-T) \\ u(t, a, \cdot) = S(T-t)u_T(T+(a-t), \cdot) + \int_t^T S(l-t)(\beta_1(l, a, \cdot)u(l, 0, \cdot) - \mu_3(l, a, \cdot)v(l, a, \cdot))dl, \\ \text{if } a \leq t + (A-T), \end{cases} \quad (2.44)$$

$$\begin{cases} v(t, a, \cdot) = \int_0^{A-a} L(A-a-l)\beta_2(t, A-l, \cdot)v(t, 0, \cdot)dl, & \text{if } a > t + (A-T) \\ v(t, a, \cdot) = L(T-t)v_T(T+(a-t), \cdot) + \int_t^T L(l-t)\beta_2(l, a, \cdot)v(l, 0, \cdot)dl, & \text{if } a \leq t + (A-T), \end{cases} \quad (2.45)$$

Thus, by the third hypothesis in (2.4) on  $\beta_1$  and  $\beta_2$  one has

$$\begin{cases} u(t, 0, \cdot) = S(T-t)u_T(T-t, \cdot) - \int_t^T S(l-t)\mu_3(l, 0, \cdot)v(l, 0, \cdot)dl, \\ v(t, 0, \cdot) = L(T-t)v_T(T+(a-t), \cdot). \end{cases} \quad (2.46)$$

Subsequently, by (2.39) we deduce that

$$\begin{aligned} & s^3 \int_Q \Theta^3 \left( \frac{x^2}{k_1(x)} u^2 e^{2s\varphi_1} + \frac{x^2}{k_2(x)} v^2 e^{2s\varphi_2} \right) dt dx + s \int_Q \Theta (k_1(x) u_x^2 e^{2s\varphi_1} + k_2(x) v_x^2 e^{2s\varphi_2}) dt dx \\ & \leq \widehat{C}_1 \left( \int_Q s^3 \Theta^3 (u^2 + v^2) e^{2s\Phi} dt dx + \int_0^1 \int_0^\delta (u_T^2(a, x) + v_T^2(a, x)) da dx \right), \end{aligned} \quad (2.47)$$

since  $(S(t))_{t \geq 0}$  and  $(L(t))_{t \geq 0}$  are a bounded semigroups,  $\mu_3 \in L^\infty(Q)$  and  $T \in (0, \delta)$ .

Then the thesis follows.  $\square$

We come now to the more challenging point and the novelty of this contribution which is the following  $\omega$ -Carleman type inequality. Such an estimate plays a crucial role to obtain the null controllability of population dynamics cascade system with one control force.

**Theorem 2.7.** *Let (2.3) and (2.4) be verified. Let  $A > 0$  and  $T > 0$  be given such that  $T \in (0, \delta)$  with  $\delta \in (0, A)$  small enough. Assume that there exists a positive constant  $\nu$  such that*

$$\mu_3 \geq \nu \quad \text{on } [0, T] \times [0, A] \times \omega_1 \quad \text{for some } \omega_1 \subseteq \omega, \quad (2.48)$$

*Then every solution  $(u, v)$  of (2.9) satisfies*

$$\begin{aligned} \int_Q \left( s^3 \Theta^3 \frac{x^2}{k_1(x)} u^2 + s \Theta k_1(x) u_x^2 \right) e^{2s\varphi_1} dt dx + \int_Q \left( s^3 \Theta^3 \frac{x^2}{k_2(x)} v^2 + s \Theta k_2(x) v_x^2 \right) e^{2s\varphi_2} dt dx \\ \leq C_\delta \left( \int_Q u^2 dt dx + \int_0^1 \int_0^\delta (u_T^2(a, x) + v_T^2(a, x)) da dx \right). \end{aligned} \quad (2.49)$$

This inequality is an immediate outcome of Theorem 2.6 applied to  $\omega_1$  and the following lemma (see for instance [7] and the references therein).

**Lemma 2.8.** *Assume that (2.3) and (2.4) hold and let  $A > 0$  and  $T > 0$  be given such that  $T \in (0, \delta)$  with  $\delta \in (0, A)$  small enough. we suppose also that (2.48) holds. Then, for all  $\epsilon > 0$  there exist two positive constants  $C$  and  $M_\epsilon$  such that for every solution  $(u, v)$  of (2.9) the following inequality is satisfied*

$$\begin{aligned} \int_{\omega_1} \int_0^A \int_0^T s^3 \Theta^3 v^2 e^{2s\Phi} dt da dx \leq \epsilon C \left( \int_Q s^3 \Theta^3 \frac{x^2}{k_2} v^2 e^{2s\varphi_2} dt da dx + \int_Q s \Theta k_2(x) v_x^2 e^{2s\varphi_2} dt da dx \right) \\ + M_\epsilon \left( \int_\omega \int_0^A \int_0^T u^2 dt da dx + \int_0^1 \int_0^\delta (u_T^2(a, x) + v_T^2(a, x)) da dx \right). \end{aligned} \quad (2.50)$$

*Proof.* Let  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  be the non-negative cut-off function defined as follows

$$\begin{cases} \chi \in \mathcal{C}^\infty(0, 1), \\ \text{supp}(\chi) \subset \omega, \\ \chi \equiv 1 \quad \text{on } \omega_1. \end{cases} \quad (2.51)$$

Recall that  $\omega = (x_1, x_2)$ . Multiplying the first equation of (2.9) by  $\chi s^3 \Theta^3 v e^{2s\Phi}$  and after an integration by parts, we get

$$\begin{aligned} \int_Q \chi s^3 \Theta^3 v e^{2s\Phi} u_t dt da dx &= - \int_Q (3 + 2s\Phi) \chi s^3 \Theta_t \Theta^2 u v e^{2s\Phi} dt da dx - \int_Q \chi s^3 \Theta^3 u v_t e^{2s\Phi} dt da dx. \\ \int_Q \chi s^3 \Theta^3 v e^{2s\Phi} u_a dt da dx &= - \int_Q (3 + 2s\Phi) \chi s^3 \Theta_a \Theta^2 u v e^{2s\Phi} dt da dx - \int_Q \chi s^3 \Theta^3 u v_a e^{2s\Phi} dt da dx. \\ \int_Q \chi s^3 \Theta^3 v e^{2s\Phi} (k_1 u_x)_x dt da dx &= - \int_Q \chi s^3 \Theta^3 k_1 e^{2s\Phi} u_x v_x dt da dx + \int_Q s^3 \Theta^3 k_1 (\chi e^{2s\Phi})_x u v_x dt da dx \\ &+ \int_Q s^3 \Theta^3 (k_1 (\chi e^{2s\Phi})_x)_x u v dt da dx. \\ - \int_Q \chi s^3 \Theta^3 v e^{2s\Phi} \mu_1 u dt da dx &= - \int_Q \chi s^3 \Theta^3 \mu_1 u v e^{2s\Phi} dt da dx. \\ - \int_Q \chi s^3 \Theta^3 v e^{2s\Phi} \mu_3 v dt da dx &= - \int_Q \chi s^3 \Theta^3 \mu_3 v^2 e^{2s\Phi} dt da dx. \end{aligned}$$

Then, summing all these identities side by side, using the second equation of (2.9) and integrating again by parts

$$\int_Q \chi s^3 \Theta^3 \mu_3 v^2 e^{2s\Phi} dt dadx = I_1 + I_2 + I_3 + I_4 + I_5, \quad (2.52)$$

where,  $I_1 := \int_Q \chi s^3 \Theta^3 \beta_1 v u(t, 0, x) e^{2s\Phi} dt dadx$ ,

$I_2 := - \int_Q ((3 + 2s\Phi) s^3 \Theta_t \Theta^2 + (3 + 2s\Phi) s^3 \Theta_a \Theta^2 + \mu_1 s^3 \Theta^3 + \mu_2 s^3 \Theta^3) \chi e^{2s\Phi} u v dt dadx$   
 $+ \int_Q s^3 \Theta^3 (k_1 (\chi e^{2s\Phi})_x)_x u v dt dadx$ ,

$I_3 := \int_Q \chi s^3 \Theta^3 \beta_2 u v(t, 0, x) e^{2s\Phi} dt dadx$ ,  $I_4 := \int_Q s^3 \Theta^3 (k_1 - k_2)(x) u v_x (\chi e^{2s\Phi})_x dt dadx$ ,

$I_5 := - \int_Q \chi s^3 \Theta^3 (k_1 + k_2)(x) u_x v_x e^{2s\Phi} dt dadx$ .

On one hand, we have by Young inequality and definition of  $\chi$

$$\begin{aligned} I_5 &\leq \epsilon \int_Q s \Theta k_2(x) v_x^2 e^{2s\varphi_2} dt dadx + \frac{1}{4\epsilon} \int_Q \frac{\chi^2 s^5 \Theta^5 (k_1 + k_2)^2 u_x^2 e^{2s(2\Phi - \varphi_2)}}{k_2} dt dadx \\ &\leq \epsilon \int_Q s \Theta k_2(x) v_x^2 e^{2s\varphi_2} dt dadx \\ &\quad + \frac{\max_{[0,1]} (k_1 + k_2)^2}{4\epsilon \min_\omega k_2} \int_Q \chi s^5 \Theta^5 u_x^2 e^{2s(2\Phi - \varphi_2)} dt dadx. \end{aligned} \quad (2.53)$$

Put  $L := \int_Q \chi s^5 \Theta^5 u_x^2 e^{2s(2\Phi - \varphi_2)} dt dadx$ . To increase  $I_5$ , we will find an upper bound of  $L$ .

To do this, we multiply the first equation of (2.9) by  $\frac{\chi s^5 \Theta^5 e^{2s(2\Phi - \varphi_2)}}{k_1} u$  and after integration by parts

$$\begin{aligned} \int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi - \varphi_2)}}{k_1} u u_t dt dadx &= -\frac{1}{2} \int_Q \frac{s^5 \chi}{k_1} \Theta^4 \Theta_t (5 + 2s(2\Phi - \varphi_2)) e^{2s(2\Phi - \varphi_2)} u^2 dt dadx. \\ \int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi - \varphi_2)}}{k_1} u u_a dt dadx &= -\frac{1}{2} \int_Q \frac{s^5 \chi}{k_1} \Theta^4 \Theta_a (5 + 2s(2\Phi - \varphi_2)) e^{2s(2\Phi - \varphi_2)} u^2 dt dadx. \\ \int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi - \varphi_2)}}{k_1} u (k_1 u_x)_x dt dadx &= - \int_Q \chi s^5 \Theta^5 u_x^2 e^{2s(2\Phi - \varphi_2)} dt dadx \\ &\quad + \frac{1}{2} \int_Q s^5 \Theta^5 \left( k_1 \left( \frac{\chi e^{2s(2\Phi - \varphi_2)}}{k_1} \right) \right)_x u^2 dt dadx. \\ - \int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi - \varphi_2)}}{k_1} u \mu_1 u dt dadx &= - \int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi - \varphi_2)}}{k_1} \mu_1 u^2 dt dadx. \\ - \int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi - \varphi_2)}}{k_1} u \mu_3 v dt dadx &= - \int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi - \varphi_2)}}{k_1} \mu_3 u v dt dadx. \end{aligned}$$

Hence, adding these equalities side by side we get

$$L = L_1 + L_2 + L_3, \quad (2.54)$$

where,  $L_1 := \int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi - \varphi_2)}}{k_1} \beta_1 u u(t, 0, x) dt dadx$ .

$L_2 := - \int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi - \varphi_2)}}{k_1} \mu_3 u v dt dadx$ .

$L_3 := - \int_Q \left( \frac{\chi s^5 \Theta^5}{k_1} \mu_1 + \frac{1}{2} \frac{s^5 \chi}{k_1} \Theta^5 \Theta_t \left( \frac{5}{\Theta} + 2s(2\Phi - \varphi_2) \right) + \frac{1}{2} \frac{s^5 \chi}{k_1} \Theta^5 \Theta_a \left( \frac{5}{\Theta} + 2s(2\Phi - \varphi_2) \right) \right) e^{2s(2\Phi - \varphi_2)} u^2 dt dadx$   
 $+ \frac{1}{2} \int_Q s^5 \Theta^5 \left( k_1 \left( \frac{\chi e^{2s(2\Phi - \varphi_2)}}{k_1} \right) \right)_x u^2 dt dadx$ .

The assumptions in (2.4) on  $\beta_1$  together with Young inequality, Lemma ??, the definitions of  $\chi$  and  $\Theta$ , the fact that the function  $x \mapsto \frac{k_2}{x^2}$  is non-increasing,  $|\Theta_t| \leq C\Theta^2$  and  $|\Theta_a| \leq \tilde{C}\Theta^2$  and

$$\sup_{(t,a,x) \in Q} s^p \Theta^p e^{2s(2\Phi-\varphi_2)} < +\infty \quad \text{for } p \in \mathbb{R}, \quad (2.55)$$

lead to

$$\begin{aligned} L_1 &\leq \frac{1}{4\epsilon} \int_Q \frac{\chi s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)}}{(k_1)^2} u^2 dt dadx + \epsilon \int_Q \chi s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)} (\beta_1)^2 u^2(t, 0, x) dt dadx \\ &\leq \frac{\tilde{K}_1}{4\epsilon} \int_Q \chi s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)} u^2 dt dadx + \epsilon K_1 \int_0^1 \int_0^A \int_{T-\delta}^T \chi u^2(t, 0, x) dt dadx \\ &\leq \frac{\tilde{K}_1}{4\epsilon} \int_Q \chi s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)} u^2 dt dadx + \epsilon K_2 \int_0^1 \int_0^\delta \chi u_T^2(a, x) dadx \end{aligned} \quad (2.56)$$

and

$$\begin{aligned} L_2 &\leq \epsilon^2 \int_Q \frac{x^2}{k_2} s^3 \Theta^3 e^{2s\varphi_2} v^2 dt dadx + \frac{1}{4\epsilon^2} \int_Q \chi^2 \frac{s^7 \Theta^7}{(k_1)^2} e^{2s(4\Phi-3\varphi_2)} \frac{k_2}{x^2} (\mu_3)^2 u^2 dt dadx \\ &\leq \epsilon^2 \int_Q \frac{x^2}{k_2} s^3 \Theta^3 e^{2s\varphi_2} v^2 dt dadx + \frac{K_4}{4\epsilon^2} \int_\omega \int_0^A \int_0^T s^7 \Theta^7 e^{2s(4\Phi-3\varphi_2)} u^2 dt dadx, \end{aligned} \quad (2.57)$$

and

$$|L_3| \leq K_5 \int_\omega \int_0^A \int_0^T s^7 \Theta^7 e^{2s(2\Phi-\varphi_2)} u^2 dt dadx, \quad (2.58)$$

where  $K_4 = \frac{\|\mu_3\|_\infty^2 k_2(x_1)}{(x_1)^2 \min_\omega k_1}$ . On the other hand, by Lemma 2.5 we have

$$e^{2s(2\Phi-\varphi_2)} \leq e^{2s(4\Phi-3\varphi_2)}. \quad (2.59)$$

Then, combining relations (2.54), (2.56), (2.57) and (2.58) we conclude

$$\begin{aligned} L &\leq \epsilon^2 \int_Q \frac{x^2}{k_2} s^3 \Theta^3 e^{2s\varphi_2} v^2 dt dadx + K_\epsilon \int_\omega \int_0^A \int_0^T s^7 \Theta^7 e^{2s(4\Phi-3\varphi_2)} u^2 dt dadx \\ &\quad + \epsilon K_2 \int_0^1 \int_0^\delta u_T^2(a, x) dadx. \end{aligned} \quad (2.60)$$

Hence, by (2.53) and (2.60) we deduce

$$\begin{aligned} I_5 &\leq \epsilon C \left( \int_Q \frac{x^2}{k_2} s^3 \Theta^3 e^{2s\varphi_2} v^2 dt dadx + \int_Q s \Theta k_2(x) v_x^2 e^{2s\varphi_2} dt dadx \right) \\ &\quad + K_\epsilon^1 \int_\omega \int_0^A \int_0^T s^7 \Theta^7 e^{2s(4\Phi-3\varphi_2)} u^2 dt dadx + K_2 \int_0^1 \int_0^\delta u_T^2(a, x) dadx. \end{aligned} \quad (2.61)$$

where  $K_\epsilon^1$  is a positive constants that depend on  $\epsilon$ . Similarly, we will find an upper bounds of  $I_1$ ,  $I_2$ ,  $I_3$  and  $I_4$ . Firstly, we will start by  $I_2$ . One has the following relations

$$\left| \int_Q \chi (3 + 2s\Phi) s^3 \Theta_t \Theta^2 e^{2s\Phi} uv dt dadx \right| \leq \int_Q \chi |3 + 2s\Phi| s^3 |\Theta_t| \Theta^2 e^{2s\Phi} |uv| dt dadx$$



$$\begin{aligned}
&\leq C \int_Q \chi |3 + 2s\Phi| s^3 \Theta^4 e^{2s\Phi} |uv| dt dx \\
&\leq \epsilon \int_Q s^3 \Theta^3 \frac{x^2}{k_2} e^{2s\varphi_2} v^2 dt dx + C_\epsilon \int_\omega \int_0^A \int_0^T s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)} u^2 dt dx, \quad (2.62)
\end{aligned}$$

$$\begin{aligned}
&\left| \int_Q \chi (3 + 2s\Phi) s^3 \Theta_a \Theta^2 e^{2s\Phi} uv dt dx \right| \\
&\leq \epsilon \int_Q s^3 \Theta^3 \frac{x^2}{k_2} e^{2s\varphi_2} v^2 dt dx + C_\epsilon^1 \int_\omega \int_0^A \int_0^T s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)} u^2 dt dx, \quad (2.63)
\end{aligned}$$

$$\begin{aligned}
&\left| \int_Q \chi (\mu_1 + \mu_2) s^3 \Theta^3 e^{2s\Phi} uv dt dx \right| \\
&\leq \epsilon \int_Q s^3 \Theta^3 \frac{x^2}{k_2} e^{2s\varphi_2} v^2 dt dx + C_\epsilon^2 \int_\omega \int_0^A \int_0^T s^3 \Theta^3 e^{2s(2\Phi-\varphi_2)} u^2 dt dx, \quad (2.64)
\end{aligned}$$

$$\begin{aligned}
&\left| \int_Q s^3 \Theta^3 (k_1 (\chi e^{2s\Phi})_x)_x uv dt dx \right| \\
&\leq \epsilon \int_Q s^3 \Theta^3 \frac{x^2}{k_2} e^{2s\varphi_2} v^2 dt dx + \frac{1}{4\epsilon} \int_Q s^3 \Theta^3 \frac{k_2}{x^2} (k_1 (\chi e^{2s\Phi})_x)_x^2 e^{-2s\varphi_2} u^2 dt dx \\
&\leq \epsilon \int_Q s^3 \Theta^3 \frac{x^2}{k_2} e^{2s\varphi_2} v^2 dt dx + \frac{C_2}{4\epsilon} \int_Q s^3 \Theta^3 \frac{k_2}{x^2} (\chi^2 + \chi_x^2 + \chi_{xx}^2) e^{2s(2\Phi-\varphi_2)} u^2 dt dx \\
&\leq \epsilon \int_Q s^3 \Theta^3 \frac{x^2}{k_2} e^{2s\varphi_2} v^2 dt dx + C_\epsilon^3 \int_\omega \int_0^A \int_0^T s^3 \Theta^3 e^{2s(2\Phi-\varphi_2)} u^2 dt dx, \quad (2.65)
\end{aligned}$$

Hence, summing inequalities (2.62), (2.63), (2.64) and (2.65) we obtain

$$I_2 \leq 4\epsilon \int_Q s^3 \Theta^3 \frac{x^2}{k_2} e^{2s\varphi_2} v^2 dt dx + C_\epsilon^4 \int_\omega \int_0^A \int_0^T s^5 \Theta^5 e^{2s(2\Phi-\varphi_2)} u^2 dt dx. \quad (2.66)$$

For the rest of integrals,

$$\begin{aligned}
I_1 &= \int_Q \chi s^3 \Theta^3 \beta_1 v u(t, 0, x) e^{2s\Phi} dt dx \\
&\leq \epsilon \int_Q s^3 \Theta^3 \frac{x^2}{k_2} e^{2s\varphi_2} v^2 dt dx + C_\epsilon^5 \int_0^1 \int_0^\delta u_T^2(a, x) da dx. \quad (2.67)
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_Q \chi s^3 \Theta^3 \beta_2 uv(t, 0, x) e^{2s\Phi} dt dx \\
&\leq \epsilon \int_0^1 \int_0^\delta v_T^2(a, x) da dx + \frac{1}{4\epsilon} \int_\omega \int_0^A \int_0^T s^7 \Theta^7 e^{2s(2\Phi-\varphi_2)} u^2 dt dx. \quad (2.68)
\end{aligned}$$

$$I_4 = \int_Q s^3 \Theta^3 (k_1 - k_2)(x) uv_x (\chi e^{2s\Phi})_x dt dx$$

$$\begin{aligned}
&= \int_Q s^3 \Theta^3 (k_1 - k_2)(x) u v_x (\chi_x + 2s \Phi_x \chi) e^{2s\Phi} dt dadx \\
&\leq \epsilon \int_Q s \Theta k_2 v_x^2 e^{2s\varphi_2} dadx + \frac{1}{4\epsilon} \int_Q s^5 \Theta^5 \frac{(k_1 - k_2)^2}{k_2} (\chi_x + 2s \Phi_x \chi)^2 e^{2s(2\Phi - \varphi_2)} u^2 dt dadx \\
&\leq \epsilon \int_Q s \Theta k_2 v_x^2 e^{2s\varphi_2} dadx + C_\epsilon^6 \int_\omega \int_0^A \int_0^T s^7 \Theta^7 e^{2s(2\Phi - \varphi_2)} u^2 dt dadx. \tag{2.69}
\end{aligned}$$

Subsequently, combining (2.61), (2.66), (2.67), (2.68), (2.69) and using again (2.59)

$$\begin{aligned}
&\int_Q \chi s^3 \Theta^3 \mu_3 v^2 e^{2s\Phi} dt dadx \leq \epsilon C_7 \left( \int_Q s^3 \Theta^3 \frac{x^2}{k_2} v^2 e^{2s\varphi_2} dt dadx + \int_Q s \Theta k_2(x) v_x^2 e^{2s\varphi_2} dt dadx \right) \\
&+ C_\epsilon^8 \int_\omega \int_0^A \int_0^T s^7 \Theta^7 e^{2s(4\Phi - 3\varphi_2)} u^2 dt dadx + C_\epsilon^9 \int_0^1 \int_0^\delta (u_T^2(a, x) + v_T^2(a, x)) dadx.
\end{aligned}$$

Finally, the hypothesis (2.48), the definition of  $\chi$  and the relation

$$\sup_{(t,a,x) \in Q} s^p \Theta^p e^{2s(4\Phi - 3\varphi_2)} < +\infty \quad \text{for } p \in \mathbb{R}, \tag{2.70}$$

yield

$$\begin{aligned}
&\int_{\omega_1} \int_0^A \int_0^T s^3 \Theta^3 v^2 e^{2s\Phi} dt dadx \leq \epsilon C_{10} \left( \int_Q s^3 \Theta^3 \frac{x^2}{k_2} v^2 e^{2s\varphi_2} dt dadx + \int_Q s \Theta k_2(x) v_x^2 e^{2s\varphi_2} dt dadx \right) \\
&+ C_\epsilon^{11} \left( \int_\omega \int_0^A \int_0^T u^2 dt dadx + \int_0^1 \int_0^\delta (u_T^2(a, x) + v_T^2(a, x)) dadx \right), \tag{2.71}
\end{aligned}$$

which finishes the proof.  $\square$

The above Carleman estimate can be used in a standard way to obtain the null controllability of the cascade system with one control force. This will be reached showing an observability inequality of the adjoint system.

### 3. Observability inequality and null controllability results

This paragraph is devoted to the observability inequality of system (2.9) and then the null controllability result of system (1.1). We start to show our observability inequality whose proof is based essentially on Carleman estimate (2.49) and Hardy-Poincaré inequality.

**Proposition 3.1.** *Assume that (2.3) and (2.4) hold. Suppose also that (2.48) is fulfilled and let  $A > 0$  and  $T > 0$  be given such that  $T \in (0, \delta)$  with  $\delta \in (0, A)$  small enough. Then, there exists a positive constant  $C_\delta$  such that for every solution  $(u, v)$  of (2.9), the following observability inequality is satisfied*

$$\int_0^1 \int_0^A (u^2(0, a, x) + v^2(0, a, x)) dadx \leq C_\delta \left( \int_q u^2 dt dadx + \int_0^1 \int_0^\delta (u_T^2(a, x) + v_T^2(a, x)) dadx \right). \tag{3.72}$$

*Proof.* Then for  $\kappa > 0$  to be defined later,  $\tilde{u} = e^{\kappa t}u$  and  $\tilde{v} = e^{\kappa t}v$  are respectively a solutions of

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} + \frac{\partial \tilde{u}}{\partial a} + (k_1(x)\tilde{u}_x)_x - \mu_1(t, a, x)\tilde{u} &= \mu_3(t, a, x)\tilde{v} - \beta_1\tilde{u}(t, 0, x) & \text{in } Q, \\ \tilde{u}(t, a, 1) = \tilde{u}(t, a, 0) &= 0 & \text{on } (0, T) \times (0, A), \\ \tilde{u}(T, a, x) &= e^{\kappa T}u_T(a, x) & \text{in } Q_A, \\ \tilde{u}(t, A, x) &= 0 & \text{in } Q_T, \end{aligned} \quad (3.73)$$

and

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial t} + \frac{\partial \tilde{v}}{\partial a} + (k_2(x)\tilde{v}_x)_x - \mu_2(t, a, x)\tilde{v} &= -\beta_2\tilde{v}(t, 0, x) & \text{in } Q, \\ \tilde{v}(t, a, 1) = \tilde{v}(t, a, 0) &= 0 & \text{on } (0, T) \times (0, A), \\ \tilde{v}(T, a, x) &= e^{\kappa T}v_T(a, x) & \text{in } Q_A, \\ \tilde{v}(t, A, x) &= 0 & \text{in } Q_T, \end{aligned} \quad (3.74)$$

where,  $u$  and  $v$  are respectively the solutions of

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + (k_1(x)u_x)_x - \mu_1(t, a, x)u &= \mu_3(t, a, x)v - \beta_1u(t, 0, x) & \text{in } Q, \\ u(t, a, 1) = u(t, a, 0) &= 0 & \text{on } (0, T) \times (0, A), \\ u(T, a, x) &= u_T(a, x) & \text{in } Q_A, \\ u(t, A, x) &= 0 & \text{in } Q_T, \end{aligned} \quad (3.75)$$

and

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{\partial v}{\partial a} + (k_2(x)v_x)_x - \mu_2(t, a, x)v &= -\beta_2v(t, 0, x) & \text{in } Q, \\ v(t, a, 1) = v(t, a, 0) &= 0 & \text{on } (0, T) \times (0, A), \\ v(T, a, x) &= v_T(a, x) & \text{in } Q_A, \\ v(t, A, x) &= 0 & \text{in } Q_T. \end{aligned} \quad (3.76)$$

Multiplying the first equations of (3.73) and (3.74) respectively by  $\tilde{u}$  and  $\tilde{v}$  and integrating by parts on  $Q_t = (0, t) \times (0, A) \times (0, 1)$  one obtains

$$\begin{aligned} &\frac{1}{2} \int_{Q_A} u^2(0, a, x)dadx + \frac{1}{2} \int_0^1 \int_0^t \tilde{u}^2(\tau, 0, x)d\tau dx \\ &+ \kappa \int_0^1 \int_0^A \int_0^t \tilde{u}^2(\tau, a, x)d\tau dadx \leq \frac{\|\beta_1\|_\infty^2 + 1}{4\epsilon'} \int_0^1 \int_0^A \int_0^t \tilde{u}^2(\tau, a, x)d\tau dadx \\ &+ \epsilon' A \int_0^1 \int_0^t \tilde{u}^2(\tau, 0, x)d\tau dx + \epsilon' \int_{Q_t} \mu_3^2 \tilde{v}^2 d\tau dadx + \frac{1}{2} \int_{Q_A} \tilde{u}^2(t, a, x)dadx. \end{aligned} \quad (3.77)$$

and

$$\begin{aligned} &\frac{1}{2} \int_{Q_A} v^2(0, a, x)dadx + \frac{1}{2} \int_0^1 \int_0^t \tilde{v}^2(\tau, 0, x)d\tau dx \\ &+ \kappa \int_0^1 \int_0^A \int_0^t \tilde{v}^2(\tau, a, x)d\tau dadx \leq \frac{\|\beta_2\|_\infty^2 + 1}{4\epsilon'} \int_0^1 \int_0^A \int_0^t \tilde{v}^2(\tau, a, x)d\tau dadx \end{aligned}$$

$$+\epsilon' A \int_0^1 \int_0^t \tilde{v}^2(\tau, 0, x) d\tau dx + \frac{1}{2} \int_{Q_A} \tilde{v}^2(t, a, x) dadx. \quad (3.78)$$

Summing (3.77) and (3.78) side by side and taking  $\kappa = \max(\frac{\|\beta_1\|_\infty^2 + 1}{4\epsilon'}, \frac{\|\beta_2\|_\infty^2 + 1}{4\epsilon'} + \epsilon' \|\mu_3\|_\infty^2)$  and  $\epsilon' < \frac{1}{2A}$ , on gets

$$\int_{Q_A} u^2(0, a, x) dadx + \int_{Q_A} v^2(0, a, x) dadx \leq \int_{Q_A} \tilde{u}^2(t, a, x) dadx + \int_{Q_A} \tilde{v}^2(t, a, x) dadx. \quad (3.79)$$

Arguing as in [2] and integrating over  $(\frac{T}{4}, \frac{3T}{4})$  we conclude

$$\begin{aligned} \int_{Q_A} u^2(0, a, x) dadx + \int_{Q_A} v^2(0, a, x) dadx &\leq C_{12} e^{2\kappa T} \left( \int_0^1 \int_0^\delta u_T^2(a, x) dadx + \int_0^1 \int_0^\delta v_T^2(a, x) dadx \right) \\ &+ \frac{2e^{2\kappa T}}{T} \left( \int_0^1 \int_\delta^A \int_{\frac{T}{4}}^{\frac{3T}{4}} u^2(t, a, x) dt dadx + \int_0^1 \int_\delta^A \int_{\frac{T}{4}}^{\frac{3T}{4}} v^2(t, a, x) dt dadx \right). \end{aligned} \quad (3.80)$$

Hence, Hardy-Poincaré inequality and the definitions of  $\varphi_i, i = 1, 2$  stated in (2.13) lead to

$$\begin{aligned} \int_{Q_A} u^2(0, a, x) dadx + \int_{Q_A} v^2(0, a, x) dadx &\leq C_{12} e^{2\kappa T} \left( \int_0^1 \int_0^\delta u_T^2(a, x) dadx + \int_0^1 \int_0^\delta v_T^2(a, x) dadx \right) \\ &+ C_\delta^{13} \left( \int_0^1 \int_\delta^A \int_{\frac{T}{4}}^{\frac{3T}{4}} s \Theta k_1(x) u^2(t, a, x) e^{2s\varphi_1} dt dadx + \int_0^1 \int_\delta^A \int_{\frac{T}{4}}^{\frac{3T}{4}} s \Theta k_2(x) v^2(t, a, x) e^{2s\varphi_2} dt dadx \right). \end{aligned}$$

Finally, using the Carleman estimate (2.49) we deduce the observability inequality (3.72). and then the proof is finished.  $\square$

Now, obtaining our observability inequality, following a standard argument, we are now ready to prove our main result.

**Theorem 3.2.** *Assume that (2.3) and (2.4) are verified. Let  $A > 0$  and  $T > 0$  be given such that  $T \in (0, \delta)$  with  $\delta \in (0, A)$  small enough. Then, for all  $(y_0, p_0) \in L^2(Q_A) \times L^2(Q_A)$ , there exists a control  $\vartheta \in L^2(q)$  such that the associated solution of (1.1) verifies*

$$\begin{cases} y(T, a, x) = 0, & \text{a.e. in } (\delta, A) \times (0, 1), \\ p(T, a, x) = 0, & \text{a.e. in } (\delta, A) \times (0, 1). \end{cases} \quad (3.81)$$

*Proof.* Let  $\varepsilon > 0$  and consider the following cost function

$$J_\varepsilon(\vartheta_1, \vartheta_2) = \frac{1}{2\varepsilon} \int_0^1 \int_\delta^A (y^2(T, a, x) + p^2(T, a, x)) dadx + \frac{1}{2} \int_q \vartheta^2(t, a, x) dt dadx.$$

We can prove that  $J_\varepsilon$  is continuous, convex and coercive. Then, it admits at least one minimizer  $\vartheta_\varepsilon$  and we have

$$\vartheta_\varepsilon = -u_\varepsilon(t, a, x) \chi_\omega(x) \quad \text{in } Q, \quad (3.82)$$

with  $u_\varepsilon$  is the solution of the following system

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial u_\varepsilon}{\partial a} + (k_1(x)(u_\varepsilon)_x)_x - \mu_1(t, a, x)u_\varepsilon - \mu_3 v_\varepsilon = -\beta_1 u_\varepsilon(t, 0, x) \quad \text{in } Q, \quad (3.83)$$

$$\begin{aligned}
u_\varepsilon(t, a, 1) &= u_\varepsilon(t, a, 0) = 0 && \text{on } (0, T) \times (0, A), \\
u_\varepsilon(T, a, x) &= \frac{1}{\varepsilon} y_\varepsilon(T, a, x) \chi_{(\delta, A)}(a) && \text{in } Q_A, \\
u_\varepsilon(t, A, x) &= 0 && \text{in } Q_T,
\end{aligned}$$

where  $v_\varepsilon$  is the solution of

$$\begin{aligned}
\frac{\partial v_\varepsilon}{\partial t} + \frac{\partial v_\varepsilon}{\partial a} + (k_2(x)(v_\varepsilon)_x)_x - \mu_2(t, a, x)v_\varepsilon &= -\beta_2 v_\varepsilon(t, 0, x) && \text{in } Q, \\
v_\varepsilon(t, a, 1) &= v_\varepsilon(t, a, 0) = 0 && \text{on } (0, T) \times (0, A), \\
v_\varepsilon(T, a, x) &= \frac{1}{\varepsilon} p_\varepsilon(T, a, x) \chi_{(\delta, A)}(a) && \text{in } Q_A, \\
v_\varepsilon(t, A, x) &= 0 && \text{in } Q_T,
\end{aligned} \tag{3.84}$$

and  $(y_\varepsilon, p_\varepsilon)$  is the solution of the system (1.1) associated to the control  $\vartheta_\varepsilon$ .

Multiplying the first equation of (3.83) by  $y_\varepsilon$  and the second equation of (1.1) by  $v_\varepsilon$ , integrating over  $Q$ , using (3.82) and the Young inequality we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon} \int_0^1 \int_\delta^A (y_\varepsilon^2(T, a, x) + p_\varepsilon^2(T, a, x)) dadx + \int_q \vartheta_\varepsilon^2(t, a, x) dt dadx \\
&= \int_{Q_A} (y_0(a, x)u_\varepsilon(0, a, x) + p_0(a, x)v_\varepsilon(0, a, x)) dadx \\
&\leq \frac{1}{4C_\delta} \int_{Q_A} (u_\varepsilon^2(0, a, x) + v_\varepsilon^2(0, a, x)) dadx + C_\delta \int_{Q_A} (y_0^2(a, x) + p_0^2(a, x)) dadx,
\end{aligned}$$

with  $C_\delta$  is the constant of the observability inequality (3.72). Hence, using relation (3.82), the observability inequality leads to

$$\begin{aligned}
&\frac{1}{\varepsilon} \int_0^1 \int_\delta^A (y_\varepsilon^2(T, a, x) + p_\varepsilon^2(T, a, x)) dadx + \frac{3}{4} \int_q \vartheta_\varepsilon^2(t, a, x) dt dadx \\
&\leq C_\delta \int_{Q_A} (y_0^2(a, x) + p_0^2(a, x)) dadx.
\end{aligned} \tag{3.85}$$

Hence, it follows that

$$\begin{cases} \int_0^1 \int_\delta^A y_\varepsilon^2(T, a, x) dadx \leq C_\delta \varepsilon \int_{Q_A} (y_0^2(a, x) + p_0(a, x)) dadx, \\ \int_0^1 \int_\delta^A p_\varepsilon^2(T, a, x) dadx \leq C_\delta \varepsilon \int_{Q_A} (y_0^2(a, x) + p_0(a, x)) dadx, \\ \int_q \vartheta_\varepsilon^2(t, a, x) dt dadx \leq \frac{4C_\delta}{3} \int_{Q_A} (y_0^2(a, x) + p_0(a, x)) dadx. \end{cases} \tag{3.86}$$

Then, we can extract two subsequences of  $(y_\varepsilon, p_\varepsilon)$  and  $\vartheta_\varepsilon$  denoted also by  $\vartheta_\varepsilon$  and  $(y_\varepsilon, p_\varepsilon)$  that converge weakly towards  $\vartheta$  and  $(y, p)$  in  $L^2(q)$  and  $L^2((0, T) \times (0, A); H_{k_1}^1(0, 1) \times H_{k_2}^1(0, 1))$  respectively. Now, by a variational technic, we prove that  $(y, p)$  is a solution of (1.1) corresponding to the controls  $\vartheta$  and, by the first and second estimates of (3.86),  $(y, p)$  satisfies (1.2).  $\square$

## 4. Appendix

As is mentioned in the introduction, this section is devoted to the proofs of some intermediate results useful to show the Carleman type inequality (2.49). Firstly, we begin by the Caccioppoli's inequality stated in the following lemma

**Lemma 4.1.** *Let  $\omega'$  be a subset of  $\omega$  such that  $\omega' \subset \subset \omega$ . Then, there exists a positive constant  $C$  such that*

$$\int_{\omega'} \int_0^A \int_0^T (u_x^2 + v_x^2) e^{2s\varphi_i} dt d\alpha dx \leq C \left( \int_Q s^2 \Theta^2 (u^2 + v^2) e^{2s\varphi_i} dt d\alpha dx + \int_Q (h_1^2 + h_2^2) e^{2s\varphi_i} dt d\alpha dx \right), \quad (4.87)$$

where  $(u, v)$  is the solution of (2.10) and the weight functions  $\varphi_i, i = 1, 2$  are defined by (2.13).

*Proof.* The proof of this result is similar to the one of [13, Lemma 5.1]. Indeed, consider the cut-off function  $\zeta$  defined by

$$\begin{cases} 0 \leq \zeta(x) \leq 1, & x \in \mathbb{R}, \\ \zeta(x) = 0, & x < x_1 \text{ and } x > x_2, \\ \zeta(x) = 1, & x \in \omega', \end{cases} \quad (4.88)$$

For  $(u, v)$  solution of (2.10) one has

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt} \left[ \int_0^1 \int_0^A \zeta^2 e^{2s\varphi_i} (u^2 + v^2) d\alpha dx \right] dt \\ &= 2s \int_0^1 \int_0^A \int_0^T \zeta^2 (\varphi_i)_t (u^2 + v^2) e^{2s\varphi_i} dt d\alpha dx + 2 \int_0^1 \int_0^A \int_0^T \zeta^2 w w_t e^{2s\varphi_i} dt d\alpha dx \\ &= 2s \int_0^1 \int_0^A \int_0^T \zeta^2 (\varphi_i)_t (u^2 + v^2) e^{2s\varphi_i} dt d\alpha dx \\ &\quad + 2 \int_0^1 \int_0^A \int_0^T \zeta^2 u (-(k_1 u_x)_x - u_a + h_1 + \mu_1 u + \mu_3 v) e^{2s\varphi_i} dt d\alpha dx \\ &\quad + 2 \int_0^1 \int_0^A \int_0^T \zeta^2 v (-(k_2 v_x)_x - v_a + h_2 + \mu_2 v) e^{2s\varphi_i} dt d\alpha dx. \end{aligned}$$

Then, integrating by parts we obtain

$$\begin{aligned} 2 \int_Q \zeta^2 (k_1 u_x^2 + k_2 v_x^2) e^{2s\varphi_i} dt d\alpha dx &= -2s \int_Q \zeta^2 (u^2 + v^2) \psi_i (\Theta_a + \Theta_t) e^{2s\varphi_i} dt d\alpha dx \\ &\quad - 2 \int_Q \zeta^2 (u h_1 + v h_2) e^{2s\varphi_i} dt d\alpha dx - 2 \int_Q \zeta^2 (\mu_1 u^2 + \mu_2 v^2) e^{2s\varphi_i} dt d\alpha dx \\ &\quad + \int_Q (k_1 (\zeta^2 e^{2s\varphi_i})_x)_x u^2 dt d\alpha dx + \int_Q (k_2 (\zeta^2 e^{2s\varphi_i})_x)_x v^2 dt d\alpha dx \\ &\quad - 2 \int_Q \zeta^2 \mu_3 u v e^{2s\varphi_i} dt d\alpha dx. \end{aligned}$$

On the other hand, by the definitions of  $\zeta$ ,  $\psi$  and  $\Theta$ , using Young inequality and taking  $s$  quite large there is a constant  $c$  such that

$$\begin{aligned}
2 \int_Q \zeta^2 (k_1 u_x^2 + k_2 v_x^2) e^{2s\varphi_i} dt dx &\geq 2 \min(\min_{x \in \omega'} k_1(x), \min_{x \in \omega'} k_2(x)) \int_{\omega'} \int_0^A \int_0^T (u_x^2 + v_x^2) e^{2s\varphi_i} dt dx, \\
\int_Q (k_1 (\zeta^2 e^{2s\varphi_i})_x)_x u^2 dt dx &\leq c \int_{\omega} \int_0^A \int_0^T s^2 \Theta^2 u^2 e^{2s\varphi_i} dt dx, \\
\int_Q (k_2 (\zeta^2 e^{2s\varphi_i})_x)_x v^2 dt dx &\leq c \int_{\omega} \int_0^A \int_0^T s^2 \Theta^2 v^2 e^{2s\varphi_i} dt dx, \\
-2s \int_Q \zeta^2 (u^2 + v^2) \psi_i (\Theta_a + \Theta_t) e^{2s\varphi_i} dt dx &\leq c \int_{\omega} \int_0^A \int_0^T s^2 \Theta^2 (u^2 + v^2) e^{2s\varphi_i} dt dx, \\
-2 \int_Q \zeta^2 u h_1 e^{2s\varphi_i} dt dx &\leq c \left( \int_{\omega} \int_0^A \int_0^T s^2 \Theta^2 u^2 e^{2s\varphi_i} dt dx + \int_{\omega} \int_0^A \int_0^T h_1^2 e^{2s\varphi_i} dt dx \right), \\
-2 \int_Q \zeta^2 v h_2 e^{2s\varphi_i} dt dx &\leq c \left( \int_{\omega} \int_0^A \int_0^T s^2 \Theta^2 v^2 e^{2s\varphi_i} dt dx + \int_{\omega} \int_0^A \int_0^T h_2^2 e^{2s\varphi_i} dt dx \right), \\
-2 \int_Q \zeta^2 (\mu_1 u^2 + \mu_2 v^2) e^{2s\varphi_i} dt dx &\leq c \int_{\omega} \int_0^A \int_0^T s^2 \Theta^2 (u^2 + v^2) e^{2s\varphi_i} dt dx, \\
-2 \int_Q \zeta^2 \mu_3 u v e^{2s\varphi_i} dt dx &\leq c \int_{\omega} \int_0^A \int_0^T s^2 \Theta^2 (u^2 + v^2) e^{2s\varphi_i} dt dx.
\end{aligned}$$

Combining all these inequalities, we can see that there is  $C > 0$  such that

$$\int_{\omega'} \int_0^A \int_0^T (u_x^2 + v_x^2) e^{2s\varphi_i} dt dx \leq C \left( \int_{\omega} \int_0^A \int_0^T s^2 \Theta^2 (u^2 + v^2) e^{2s\varphi_i} dt dx + \int_{\omega} (h_1^2 + h_2^2) e^{2s\varphi_i} dt dx \right).$$

Thus, the proof is achieved.  $\square$

**Remark 4.2.** In Lemma 4.1, the set  $\omega'$  is chosen so that 0 which is exactly the point of degeneracy of the dispersion coefficients  $k_1$  and  $k_2$  does not belong to  $\overline{\omega'}$ . More generally, if the degeneracy occurs at a point  $x_0 \in (0, 1)$ , one must take  $x_0$  out of  $\overline{\omega'}$  in the case of interior degeneracy to establish a Caccioppoli's type inequality (see [10] for more details in this context).

We close this section by the following result

**Lemma 4.3.** Assume that the conditions (2.15) hold. Then,  $I = [\frac{k_2(1)(2-\gamma)(e^{2\kappa\|\sigma\|_\infty} - 1)}{d_2 k_2(1)(2-\gamma) - 1}, \frac{4(e^{2\kappa\|\sigma\|_\infty} - e^{\kappa\|\sigma\|_\infty})}{3d_2})$  is not empty.

*Proof.* Indeed, one has

$$\begin{aligned}
&\frac{4(e^{2\kappa\|\sigma\|_\infty} - e^{\kappa\|\sigma\|_\infty})}{3d_2} - \frac{k_2(1)(2-\gamma)(e^{2\kappa\|\sigma\|_\infty} - 1)}{d_2 k_2(1)(2-\gamma) - 1} \\
&= \frac{4(e^{2\kappa\|\sigma\|_\infty} - e^{\kappa\|\sigma\|_\infty})(d_2 k_2(1)(2-\gamma) - 1) - 3d_2 k_2(1)(2-\gamma)(e^{2\kappa\|\sigma\|_\infty} - 1)}{3d_2(d_2 k_2(1)(2-\gamma) - 1)} \\
&= \frac{e^{2\kappa\|\sigma\|_\infty}(d_2 k_2(1)(2-\gamma) - 4) - 4e^{\kappa\|\sigma\|_\infty}(d_2 k_2(1)(2-\gamma) - 1)}{3d_2(d_2 k_2(1)(2-\gamma) - 1)} + \frac{k_2(1)(2-\gamma)}{d_2 k_2(1)(2-\gamma) - 1}
\end{aligned}$$



$$= \frac{e^{\kappa\|\sigma\|_\infty} [e^{\kappa\|\sigma\|_\infty} (d_2 k_2(1)(2-\gamma) - 4) - 4(d_2 k_2(1)(2-\gamma) - 1)]}{3d_2(d_2 k_2(1)(2-\gamma) - 1)} + \frac{k_2(1)(2-\gamma)}{d_2 k_2(1)(2-\gamma) - 1}.$$

Using the fact that  $d_2 \geq \frac{5}{k_2(1)(2-\gamma)}$ , we can conclude that  $\frac{4(d_2 k_2(1)(2-\gamma)-1)}{d_2 k_2(1)(2-\gamma)-4} \leq 16$ .

Since  $\kappa \geq \frac{4 \ln 2}{\|\sigma\|_\infty}$ , then we have  $e^{\kappa\|\sigma\|_\infty} \geq 16$ . Therefore, the previous difference is positive and subsequently  $I \neq \emptyset$ .  $\square$

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